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Composite schemes for multivariate blending rational interpolation ☆

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Abstract

It is demonstrated that Newton's interpolation polynomials and Thiele's interpolating continued fractions can be incorporated to generate various interpolation schemes based on rectangular grids, among them are two kinds of bivariate blending rational interpolants. However, blending rational interpolants strongly depend on the existence of so-called blending differences, which means that for some grids of data, one may fail to find out the corresponding rational interpolants as a whole. In this paper, we offer a solution scheme by adopting composite interpolation over triangular sub-grids. Characterization theorem is given, error estimation is worked out and vector valued case as well as matrix valued case is discussed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

As is well known, a function $f(x)$ defined on some set G can be approximated by both Newton's interpolating polynomials

$$N_n(x)=\sum_{i=0}^{n} f[x_0,x_1,\ldots,x_i]\prod_{j=0}^{i-1}(x-x_j)$$

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and Thiele's interpolating continued fractions

$$T_n(x) = \varphi[x_0] + \sum_{i=1}^{n} \frac{x - x_{i-1}}{\varphi[x_0, x_1, \dots, x_i]},$$

where $x_0, x_1, \dots, x_n$ are the points contained in G. $f[x_0, x_1, \dots, x_i]$ is the divided difference of $f(x)$ at $x_0, x_1, \dots, x_i$ and $\varphi[x_0, x_1, \dots, x_i]$ denotes the inverse difference of $f(x)$ at $x_0, x_1, \dots, x_i$. It goes without saying that the key parts of the above mentioned interpolants are the divided differences and the inverse differences which allow the interpolants to be computed recursively.

For simplicity and also without loss of generality, we only restrict ourselves to the case where bivariate problems are involved.

2. Bivariate blending rational interpolants

Suppose

$$\Pi_{m,n} = \{(x_i, y_j) \mid i = 0, 1, \dots, m;\ j = 0, 1, \dots, n\}$$

is a planar set of points and $f(x, y)$ is a bivariate function defined on the domain D enclosing $\Pi_{m,n}$.

Definition 1. Let

$$\varphi[x, y] = f(x, y) \quad \forall (x, y) \in D, \tag{2.1}$$

$$\varphi[x_i; y_0, \dots, y_j] = \frac{\varphi[x_i; y_0, \dots, y_{j-2}, y_j] - \varphi[x_i; y_0, \dots, y_{j-1}]}{y_j - y_{j-1}}, \tag{2.2}$$

$$\varphi[x_0, \dots, x_i; y_j] = \frac{x_i - x_{i-1}}{\varphi[x_0, \dots, x_{i-2}, x_i; y_j] - \varphi[x_0, \dots, x_{i-1}; y_j]}, \tag{2.3}$$

$$\varphi[x_0, \dots, x_i; y_0, \dots, y_j] = \frac{\varphi[x_0, \dots, x_i; y_0, \dots, y_{j-2}, y_j] - \varphi[x_0, \dots, x_i; y_0, \dots, y_{j-1}]}{y_j - y_{j-1}}. \tag{2.4}$$

Then $\varphi[x_0, \dots, x_i; y_0, \dots, y_j]$ is called the blending difference of Thiele–Newton type of $f(x, y)$ at the set of points $\{x_0, \dots, x_i\} \times \{y_0, \dots, y_j\}$.

Definition 2. Let

$$\psi[x, y] = f(x, y) \quad \forall (x, y) \in D, \tag{2.5}$$

$$\psi[x_i; y_0, \dots, y_j] = \frac{y_j - y_{j-1}}{\psi[x_i; y_0, \dots, y_{j-2}, y_j] - \psi[x_i; y_0, \dots, y_{j-1}]}, \tag{2.6}$$

$$\psi[x_0,\dots,x_i;y_j]=\frac{\psi[x_0,\dots,x_{i-2},x_i;y_j]-\psi[x_0,\dots,x_{i-1};y_j]}{x_i-x_{i-1}}, \tag{2.7}$$

$$\psi[x_0,\dots,x_i;y_0,\dots,y_j] = \frac{y_j-y_{j-1}}{\psi[x_0,\dots,x_i;y_0,\dots,y_{j-2},y_j]-\psi[x_0,\dots,x_i;y_0,\dots,y_{j-1}]}. \tag{2.8}$$

Then $\psi[x_0,\dots,x_i;y_0,\dots,y_j]$ is called the blending difference of Newton–Thiele type of $f(x,y)$ at the set of points $\{x_0,\dots,x_i\}\times\{y_0,\dots,y_j\}$.

With the blending differences of Thiele–Newton type, we can construct the following blending rational interpolants

$$\mathrm{TN}_{m,n}(x,y)=N_0(y)+\frac{x-x_0|}{|N_1(y)}+\frac{x-x_1|}{|N_2(y)}+\cdots+\frac{x-x_{m-1}|}{|N_m(y)}, \tag{2.9}$$

where for $i=0,1,\dots,m$

$$N_i(y)=\varphi[x_0,\dots,x_i;y_0]+(y-y_0)\varphi[x_0,\dots,x_i;y_0,y_1] + \cdots+(y-y_0)\cdots(y-y_{n-1})\varphi[x_0,\dots,x_i;y_0,\dots,y_n] \tag{2.10}$$

while with the blending differences of Newton–Thiele type, we can establish the blending rational interpolants of the following form

$$\mathrm{NT}_{m,n}(x,y)=T_0(y)+(x-x_0)T_1(y)+\cdots+(x-x_0)\cdots(x-x_{m-1})T_m(y), \tag{2.11}$$

where for $i=0,1,\dots,m$

$$T_i(y)=\psi[x_0,\dots,x_i;y_0]+\frac{y-y_0|}{|\psi[x_0,\dots,x_i;y_0,y_1]}+\cdots+\frac{y-y_{n-1}|}{|\psi[x_0,\dots,x_i;y_0,\dots,y_n]}. \tag{2.12}$$

It is not difficult to prove (see [2,4])

$$\mathrm{TN}_{m,n}(x_i,y_j)=\mathrm{NT}_{m,n}(x_i,y_j)=f(x_i,y_j)\quad \forall (x_i,y_j)\in\Pi_{m,n}. \tag{2.13}$$

For brevity, the blending rational interpolants defined by (2.9) and (2.11) will hereafter be cited as TNBRIs and NTBRIs, respectively.

3. Composite schemes

Assume $m=n$ and let $\Pi_{n,n}$ be displayed in the following square grid:

$$\begin{array}{cccc}
(x_0,y_0) & (x_1,y_0) & \cdots & (x_n,y_0)\\
(x_0,y_1) & (x_1,y_1) & \cdots & (x_n,y_1)\\
\vdots & \vdots & \ddots & \vdots\\
(x_0,y_n) & (x_1,y_n) & \cdots & (x_n,y_n)
\end{array}$$

Then, we may divide the square grid into two subgrids, i.e., the lower triangular grid

$$\begin{array}{llll}
(x_0, y_0) & & & \\
(x_0, y_1) & (x_1, y_1) & & \\
\vdots & \vdots & \ddots & \\
(x_0, y_n) & (x_1, y_n) & \cdots & (x_n, y_n)
\end{array}$$

and the upper triangular grid

$$\begin{array}{llll}
(x_1, y_0) & (x_2, y_0) & \cdots & (x_n, y_0) \\
 & (x_2, y_1) & \cdots & (x_n, y_1) \\
 & & \ddots & \vdots \\
 & & & (x_n, y_{n-1})
\end{array}$$

Denote by S_{L} and S_{U} the lower triangular grid and the upper triangular grid, respectively, namely,

$$S_{\mathrm{L}} = \{(x_i, y_j) \mid i = 0, 1, \ldots, n;\ j = i, i+1, \ldots, n\},$$
$$S_{\mathrm{U}} = \{(x_i, y_j) \mid i = 1, 2, \ldots, n;\ j = 0, 1, \ldots, i-1\}.$$

Then with S_{L} one may construct the blending rational interpolant $\mathrm{TN}_n^{\mathrm{L}}(x, y)$ of Thiele–Newton type and the blending rational interpolant $\mathrm{NT}_n^{\mathrm{L}}(x, y)$ of Newton–Thiele type as follows:

$$\mathrm{TN}_n^{\mathrm{L}}(x, y) = N_0^{\mathrm{L}}(y) + \frac{x - x_0}{N_1^{\mathrm{L}}(y)} + \frac{x - x_1}{N_2^{\mathrm{L}}(y)} + \cdots + \frac{x - x_{n-1}}{N_n^{\mathrm{L}}(y)}, \tag{3.1}$$

where for $i = 0, 1, \ldots, n$

$$N_i^{\mathrm{L}}(y) = a_{i,i} + a_{i,i+1}(y - y_i) + a_{i,i+2}(y - y_i)(y - y_{i+1}) + \cdots + a_{i,n}(y - y_i) \cdots (y - y_{n-1}). \tag{3.2}$$

$$\mathrm{NT}_n^{\mathrm{L}}(x, y) = T_0^{\mathrm{L}}(y) + T_1^{\mathrm{L}}(y)(x - x_0) + \cdots + T_n^{\mathrm{L}}(y)(x - x_0) \cdots (x - x_{n-1}), \tag{3.3}$$

where for $i = 0, 1, \ldots, n$

$$T_i^{\mathrm{L}}(y) = b_{i,i} + \frac{y - y_i}{b_{i,i+1}} + \frac{y - y_{i+1}}{b_{i,i+2}} + \cdots + \frac{y - y_{n-1}}{b_{i,n}}. \tag{3.4}$$

With S_{U} one may construct the blending rational interpolant $\mathrm{TN}_n^{\mathrm{U}}(x, y)$ of Thiele–Newton type and the blending interpolant $\mathrm{NT}_n^{\mathrm{U}}(x, y)$ of Newton–Thiele type as follows:

$$\mathrm{TN}_n^{\mathrm{U}}(x, y) = N_0^{\mathrm{U}}(y) + \frac{x - x_n}{N_1^{\mathrm{U}}(y)} + \frac{x - x_{n-1}}{N_2^{\mathrm{U}}(y)} + \cdots + \frac{x - x_2}{N_{n-1}^{\mathrm{U}}(y)}, \tag{3.5}$$

where for $i=0,1,\ldots,n-1$

$$N_i^{\mathrm{U}}(y)=c_{i,0}+c_{i,1}(y-y_0)+c_{i,2}(y-y_0)(y-y_1)+\cdots+c_{i,n-i-1}(y-y_0)\cdots(y-y_{n-i-2}). \tag{3.6}$$

$$\mathrm{NT}_n^{\mathrm{U}}(x,y)=T_0^{\mathrm{U}}(y)+T_1^{\mathrm{U}}(y)(x-x_n)+\cdots+T_{n-1}^{\mathrm{U}}(y)(x-x_n)(x-x_{n-1})\cdots(x-x_2), \tag{3.7}$$

where for $i=0,1,\ldots,n-1$

$$T_i^{\mathrm{U}}(y)=d_{i,0}+\frac{y-y_0}{d_{i,1}}+\frac{y-y_1}{d_{i,2}}+\cdots+\frac{y-y_{n-i-2}}{d_{i,n-i-1}}. \tag{3.8}$$

Let

$$v_{i,j}=\begin{cases} f(x_i,y_j)/U(x_i,y_j) & \text{for } (x_i,y_j)\in S_{\mathrm{L}},\\ f(x_i,y_j)/L(x_i,y_j) & \text{for } (x_i,y_j)\in S_{\mathrm{U}},\end{cases} \tag{3.9}$$

where $L(x,y)$ and $U(x,y)$ are polynomials such that

$$L(x_i,y_j)=0 \quad \forall(x_i,y_j)\in S_{\mathrm{L}},$$

$$L(x_i,y_j)\neq 0 \quad \forall(x_i,y_j)\in S_{\mathrm{U}},$$

$$U(x_i,y_j)=0 \quad \forall(x_i,y_j)\in S_{\mathrm{U}},$$

$$U(x_i,y_j)\neq 0 \quad \forall(x_i,y_j)\in S_{\mathrm{L}}.$$

Theorem 1. *Let*

$$a_{i,j}=\varphi[x_0,\ldots,x_i;y_i,\ldots,y_j],$$

where $\varphi[x_0,\ldots,x_i;y_i,\ldots,y_j]$ *is recursively defined in* (2.2)–(2.4) *and* $\varphi[x_p;y_q]=v_{p,q}$, $\forall(x_p,y_q)\in S_{\mathrm{L}}$, *then* $\mathrm{TN}_n^{\mathrm{L}}(x,y)$ *defined in* (3.1) *and* (3.2) *satisfies*

$$\mathrm{TN}_n^{\mathrm{L}}(x_i,y_j)=v_{i,j} \quad \forall(x_i,y_j)\in S_{\mathrm{L}}.$$

Proof. From (2.2)–(2.4), it follows:

$$\begin{aligned} N_i^{\mathrm{L}}(y_j)&=\varphi[x_0,\ldots,x_i;y_i]+\varphi[x_0,\ldots,x_i;y_i,y_{i+1}](y_j-y_i)\\ &\quad+\cdots+\varphi[x_0,\ldots,x_i;y_i,\ldots,y_j](y_j-y_i)\cdots(y_j-y_{j-1})\\ &=\varphi[x_0,\ldots,x_i;y_i]+\varphi[x_0,\ldots,x_i;y_i,y_{i+1}](y_j-y_i)\\ &\quad+\cdots+\varphi[x_0,\ldots,x_i;y_i,\ldots,y_{j-2},y_j](y_j-y_i)\cdots(y_j-y_{j-2})\\ &=\cdots\\ &=\varphi[x_0,\ldots,x_i;y_i]+\varphi[x_0,\ldots,x_i;y_i,y_j](y_j-y_i)\\ &=\varphi[x_0,\ldots,x_i;y_j]\end{aligned}$$

which leads to

$$\begin{aligned}
\mathrm{TN}_n^{\mathrm{L}}(x_i,y_j) &= \varphi[x_0;y_j] + \frac{x_i-x_0}{\varphi[x_0,x_1;y_j]} + \cdots + \frac{x_i-x_{i-1}}{\varphi[x_0,\ldots,x_i;y_j]} \\
&= \varphi[x_0;y_j] + \frac{x_i-x_0}{\varphi[x_0,x_1;y_j]} + \cdots + \frac{x_i-x_{i-2}}{\varphi[x_0,\ldots,x_{i-2},x_i;y_j]} \\
&= \cdots \\
&= \varphi[x_0;y_j] + \frac{x_i-x_0}{\varphi[x_0,x_i;y_j]} \\
&= \varphi[x_i;y_j] \\
&= v_{i,j} \quad \forall (x_i,y_j)\in S_{\mathrm{L}}
\end{aligned}$$

as asserted.

Similarly, one can prove the following theorems:

Theorem 2. *Let*

$$b_{i,j} = \psi[x_0,\ldots,x_i;y_i,\ldots,y_j],$$

where $\psi[x_0,\ldots,x_i;y_i,\ldots,y_j]$ *is recursively defined in* (2.6)–(2.8) *and* $\psi[x_p;y_q]=v_{p,q}$, $\forall(x_p,y_q)\in S_{\mathrm{L}}$, *then* $\mathrm{NT}_n^{\mathrm{L}}(x,y)$ *defined in* (3.3) *and* (3.4) *satisfies*

$$\mathrm{NT}_n^{\mathrm{L}}(x_i,y_j) = v_{i,j} \quad \forall (x_i,y_j)\in S_{\mathrm{L}}.$$

Theorem 3. *Let*

$$c_{i,j} = \varphi[x_n,x_{n-1},\ldots,x_{n-i};y_0,\ldots,y_j],$$

where $\varphi[x_n,x_{n-1},\ldots,x_{n-i};y_0,\ldots,y_j]$ *is recursively determined by* (2.2)–(2.4) *and* $\varphi[x_p;y_q]=v_{p,q}$, $\forall(x_p,y_q)\in S_{\mathrm{U}}$, *then* $\mathrm{TN}_n^{\mathrm{U}}(x,y)$ *defined in* (3.5) *and* (3.6) *satisfies*

$$\mathrm{TN}_n^{\mathrm{U}}(x_i,y_j) = v_{i,j} \quad \forall (x_i,y_j)\in S_{\mathrm{U}}.$$

Theorem 4. *Let*

$$d_{i,j} = \psi[x_n,x_{n-1},\ldots,x_{n-i};y_0,\ldots,y_j],$$

where $\psi[x_n,x_{n-1},\ldots,x_{n-i};y_0,\ldots,y_j]$ *is recursively determined by* (2.6)–(2.8) *and* $\psi[x_p;y_q]=v_{p,q}$, $\forall(x_p,y_q)\in S_{\mathrm{U}}$, *then* $\mathrm{NT}_n^{\mathrm{U}}(x,y)$ *defined in* (3.7) *and* (3.8) *satisfies*

$$\mathrm{NT}_n^{\mathrm{U}}(x_i,y_j) = v_{i,j} \quad \forall (x_i,y_j)\in S_{\mathrm{U}}.$$

Now we can assemble the four blending rational interpolants $\mathrm{TN}_n^{\mathrm{L}}(x,y)$, $\mathrm{NT}_n^{\mathrm{L}}(x,y)$, $\mathrm{TN}_n^{\mathrm{U}}(x,y)$ and $\mathrm{NT}_n^{\mathrm{U}}(x,y)$ as formulated in Theorems 1–4, respectively, into four composite schemes as follows:

$$\mathrm{TNTN}_n(x,y) = U(x,y)\mathrm{TN}_n^{\mathrm{L}}(x,y) + L(x,y)\mathrm{TN}_n^{\mathrm{U}}(x,y), \tag{3.10}$$

$$\mathrm{TNNT}_n(x,y)=U(x,y)\mathrm{TN}_n^{\mathrm{L}}(x,y)+L(x,y)\mathrm{NT}_n^{\mathrm{U}}(x,y), \tag{3.11}$$

$$\mathrm{NTTN}_n(x,y)=U(x,y)\mathrm{NT}_n^{\mathrm{L}}(x,y)+L(x,y)\mathrm{TN}_n^{\mathrm{U}}(x,y), \tag{3.12}$$

$$\mathrm{NTNT}_n(x,y)=U(x,y)\mathrm{NT}_n^{\mathrm{L}}(x,y)+L(x,y)\mathrm{NT}_n^{\mathrm{U}}(x,y). \tag{3.13}$$

From (3.9), it follows:

$$\mathrm{TNTN}_n(x_i,y_j)=f(x_i,y_j) \quad \forall (x_i,y_j)\in \Pi_{n,n}$$

$$\mathrm{TNNT}_n(x_i,y_j)=f(x_i,y_j) \quad \forall (x_i,y_j)\in \Pi_{n,n}$$

$$\mathrm{NTTN}_n(x_i,y_j)=f(x_i,y_j) \quad \forall (x_i,y_j)\in \Pi_{n,n}$$

$$\mathrm{NTNT}_n(x_i,y_j)=f(x_i,y_j) \quad \forall (x_i,y_j)\in \Pi_{n,n}.$$

4. Characterization theorem

Denote by d_x^0P and d_y^0P the degree of the polynomial $P(x,y)$ with respect to variables x and y, respectively.

Definition 3. Suppose $R(x,y)=P(x,y)/Q(x,y)$, then we say the rational function $R(x,y)$ is of type (d_x^0P/d_x^0Q) with respect to x and of type (d_y^0P/d_y^0Q) with respect to y.

Theorem 5. *For $k=0,1,\dots,n$, let*

$$\frac{P_k^{\mathrm{L}}(x,y)}{Q_k^{\mathrm{L}}(x,y)}=N_0^{\mathrm{L}}(y)+\frac{x-x_0}{N_1^{\mathrm{L}}(y)}+\cdots+\frac{x-x_{k-1}}{N_k^{\mathrm{L}}(y)},$$

then

$$d_x^0P_k^{\mathrm{L}}(x,y)=\left[\frac{k+1}{2}\right], \quad d_x^0Q_k^{\mathrm{L}}(x,y)=\left[\frac{k}{2}\right],$$

$$d_y^0P_k^{\mathrm{L}}(x,y)=\frac{(k+1)(2n-k)}{2}, \quad d_y^0Q_k^{\mathrm{L}}(x,y)=\frac{k(2n-k-1)}{2},$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From (3.2), it follows:

$$d_y^0N_i^{\mathrm{L}}(y)=n-i, \quad d_x^0N_i^{\mathrm{L}}(y)=0, \ \ i=0,1,\dots,n.$$

Since

$$\frac{P_0^{\mathrm{L}}(x,y)}{Q_0^{\mathrm{L}}(x,y)}=N_0^{\mathrm{L}}(y)$$

and

$$\frac{P_1^{\mathrm{L}}(x,y)}{Q_1^{\mathrm{L}}(x,y)} = N_0^{\mathrm{L}}(y) + \frac{x-x_0}{N_1^{\mathrm{L}}(y)} = \frac{N_0^{\mathrm{L}}(y)N_1^{\mathrm{L}}(y) + x - x_0}{N_1^{\mathrm{L}}(y)},$$

one gets

$$d_x^0 P_0^{\mathrm{L}}(x,y) = 0, \quad d_x^0 Q_0^{\mathrm{L}}(x,y) = 0,$$
$$d_y^0 P_0^{\mathrm{L}}(x,y) = n, \quad d_y^0 Q_0^{\mathrm{L}}(x,y) = 0,$$
$$d_x^0 P_1^{\mathrm{L}}(x,y) = 1, \quad d_x^0 Q_1^{\mathrm{L}}(x,y) = 0,$$
$$d_y^0 P_1^{\mathrm{L}}(x,y) = 2n-1, \quad d_y^0 Q_1^{\mathrm{L}}(x,y) = n-1.$$

Therefore, the conclusion of Theorem 5 holds true for $k=0,1$. Now assume it is valid for $k=0, 1,\ldots,m$, $1 \leqslant m < n$, then by the recursive formula of continued fraction, one has

$$P_{m+1}^{\mathrm{L}}(x,y) = N_{m+1}^{\mathrm{L}}(y)P_m^{\mathrm{L}}(x,y) + (x-x_m)P_{m-1}^{\mathrm{L}}(x,y),$$
$$Q_{m+1}^{\mathrm{L}}(x,y) = N_{m+1}^{\mathrm{L}}(y)Q_m^{\mathrm{L}}(x,y) + (x-x_m)Q_{m-1}^{\mathrm{L}}(x,y)$$

which implies

$$\begin{aligned}
d_x^0 P_{m+1}^{\mathrm{L}}(x,y) &= \max\{d_x^0 N_{m+1}^{\mathrm{L}}(y) + d_x^0 P_m^{\mathrm{L}}(x,y), 1 + d_x^0 P_{m-1}^{\mathrm{L}}(x,y)\} \\
&= \max\left\{\left[\frac{m+1}{2}\right], 1 + \left[\frac{m}{2}\right]\right\} \\
&= \left[\frac{m+2}{2}\right], \\
d_x^0 Q_{m+1}^{\mathrm{L}}(x,y) &= \max\{d_x^0 N_{m+1}^{\mathrm{L}}(y) + d_x^0 Q_m^{\mathrm{L}}(x,y), 1 + d_x^0 Q_{m-1}^{\mathrm{L}}(x,y)\} \\
&= \max\left\{\left[\frac{m}{2}\right], 1 + \left[\frac{m-1}{2}\right]\right\} \\
&= \left[\frac{m+1}{2}\right], \\
d_y^0 P_{m+1}^{\mathrm{L}}(x,y) &= \max\{d_y^0 N_{m+1}^{\mathrm{L}}(y) + d_y^0 P_m^{\mathrm{L}}(x,y), d_y^0 P_{m-1}^{\mathrm{L}}(x,y)\} \\
&= \max\left\{n-m-1 + \frac{(m+1)(2n-m)}{2}, \frac{m(2n-m+1)}{2}\right\} \\
&= \frac{(m+2)(2n-m-1)}{2},
\end{aligned}$$

$$\begin{aligned}d_y^0 Q_{m+1}^{\mathrm{L}}(x,y) &= \max\{d_y^0 N_{m+1}^{\mathrm{L}}(y) + d_y^0 Q_m^{\mathrm{L}}(x,y), d_y^0 Q_{m-1}^{\mathrm{L}}(x,y)\}\\ &= \max\left\{n-m-1+\frac{m(2n-m-1)}{2}, \frac{(m-1)(2n-m)}{2}\right\}\\ &= \frac{(m+1)(2n-m-2)}{2}.\end{aligned}$$

Therefore, the conclusion is valid for $n=m+1$. Thus, Theorem 5 is proved by induction.

By Theorem 5, it is easy to know that the blending rational interpolant $\mathrm{TN}_n^{\mathrm{L}}(x,y)$ is of type $([(n+1)/2]/[n/2])$ with respect to x and of type $((n(n+1)/2)/(n(n-1)/2))$ with respect to y while the blending rational interpolant $\mathrm{TN}_n^{\mathrm{U}}(x,y)$ is of type $([n/2]/[(n-1)/2])$ with respect to x and of type $((n(n-1)/2)/((n-1)(n-2)/2))$ with respect to y. Moreover with the help of the following summation formula

$$\sum_{i=0}^{n}\left[\frac{i}{2}\right] = \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right],$$

we can also draw the conclusion that the blending rational interpolant $\mathrm{NT}_n^{\mathrm{L}}(x,y)$ is of type $(n/0)$ with respect to x and of type

$$\left(\left[\frac{n-1}{2}\right]\left[\frac{n}{2}\right]+\left[\frac{n+1}{2}\right]\Big/\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]\right)$$

with respect to y while the blending rational interpolant $\mathrm{NT}_n^{\mathrm{U}}(x,y)$ is of type $(n-1/0)$ with respect to x and of type

$$\left(\left[\frac{n-2}{2}\right]\left[\frac{n-1}{2}\right]+\left[\frac{n}{2}\right]\Big/\left[\frac{n-1}{2}\right]\left[\frac{n}{2}\right]\right)$$

with respect to y.

5. Error estimation

We turn now to a discussion of the error in the approximation of a function $f(x,y)$ by its composite blending rational interpolants.

Theorem 6. *Suppose D is a domain containing $\Pi_{n,n}$ and the function $f(x,y)$ is differentiable in D up to $(n+1)$ times. Let*

$$V(x,y) = \frac{f(x,y)}{L(x,y)+U(x,y)},$$

$$\mathrm{TN}_n^{\mathrm{L}}(x,y) = \frac{P_n^{\mathrm{L}}(x,y)}{Q_n^{\mathrm{L}}(x,y)},$$

$$\mathrm{TN}_n^{\mathrm{U}}(x,y) = \frac{P_n^{\mathrm{U}}(x,y)}{Q_n^{\mathrm{U}}(x,y)},$$

$$F(x,y) = Q_n^{\mathrm{L}}(x,y)\,[V(x,y) - \mathrm{TN}_n^{\mathrm{L}}(x,y)],$$

$$G(x,y) = Q_n^{\mathrm{U}}(x,y)\,[V(x,y) - \mathrm{TN}_n^{\mathrm{U}}(x,y)],$$

$$\omega_i(x) = (x - x_0)(x - x_1)\cdots(x - x_{i-1}),$$

$$\bar{\omega}_j(y) = (y - y_0)(y - y_1)\cdots(y - y_{j-1})$$

and denote by $I[x_0, x_1, \ldots, x_i]$ *the smallest interval containing* $x_0, x_1, \ldots, x_i$, *then for each pair of arguments* $(x,y) \in D$

$$\begin{aligned} &f(x,y) - \mathrm{TNTN}_n(x,y) \\ &\quad = \frac{U(x,y)}{Q_n^{\mathrm{L}}(x,y)} \left[\frac{\bar{\omega}_{n+1}(y)}{(n+1)!} \sum_{i=0}^{n} \binom{n+1}{i} \frac{\omega_i(x)}{\bar{\omega}_i(y)} \frac{\partial^{n+1} F(\xi_i,\eta_i)}{\partial x^i \partial y^{n-i+1}} + \frac{\omega_{n+1}(x)}{(n+1)!} \frac{\partial^{n+1} F(\xi,y)}{\partial x^{n+1}} \right] \\ &\qquad + \frac{L(x,y)}{Q_n^{\mathrm{U}}(x,y)} \left[\frac{\omega_{n+1}(x)}{n!} \sum_{i=2}^{n+1} \binom{n}{i-1} \frac{\bar{\omega}_{i-1}(y)}{\omega_i(x)} \frac{\partial^n G(\lambda_i,\theta_i)}{\partial x^{n-i+1} \partial y^{i-1}} + \frac{\omega_{n+1}(x)}{n!\,\omega_1(x)} \frac{\partial^n G(\lambda,y)}{\partial x^n} \right], \end{aligned}$$

where

$$\xi_i \in I[x_0, x_1, \ldots, x_i], \quad i = 0, 1, \ldots, n,$$

$$\eta_i \in I[y_i, y_{i+1}, \ldots, y_n, y], \quad i = 0, 1, \ldots, n,$$

$$\xi \in I[x_0, x_1, \ldots, x_n, x],$$

$$\lambda_i \in I[x_{i-1}, x_i, \ldots, x_n], \quad i = 2, 3, \ldots, n+1,$$

$$\theta_i \in I[y_0, y_1, \ldots, y_{i-2}, y], \quad i = 2, 3, \ldots, n+1,$$

$$\lambda \in I[x_1, x_2, \ldots, x_n, x].$$

Proof. From

$$V(x,y) = \frac{f(x,y)}{L(x,y) + U(x,y)},$$

$$\mathrm{TN}_n^{\mathrm{L}}(x,y) = P_n^{\mathrm{L}}(x,y)/Q_n^{\mathrm{L}}(x,y)$$

and

$$\mathrm{TN}_n^{\mathrm{U}}(x,y) = P_n^{\mathrm{U}}(x,y)/Q_n^{\mathrm{U}}(x,y)$$

it follows by Theorems 1 and 3

$$F(x_i, y_j) = 0 \quad \forall (x_i, y_j) \in S_{\mathrm{L}},$$

$$G(x_i, y_j) = 0 \quad \forall (x_i, y_j) \in S_{\mathrm{U}}.$$

Making use of the Newton interpolation formula (see [5]), we have

$$F(x,y)=\sum_{i=0}^{n}\omega_i(x)F[x_0,\dots,x_i;y]+\omega_{n+1}(x)F[x_0,\dots,x_n,x;y]$$

$$=\sum_{i=0}^{n}\omega_i(x)\left(\sum_{j=i}^{n}\frac{\bar{\omega}_j(y)}{\bar{\omega}_i(y)}F[x_0,\dots,x_i;y_i,\dots,y_j]\right.$$

$$\left.+\ \frac{\bar{\omega}_{n+1}(y)}{\bar{\omega}_i(y)}F[x_0,\dots,x_i;y_i,\dots,y_n,y]\right)$$

$$+\,\omega_{n+1}(x)F[x_0,\dots,x_n,x;y]$$

$$=\bar{\omega}_{n+1}(y)\sum_{i=0}^{n}\frac{\omega_i(x)}{\bar{\omega}_i(y)}F[x_0,\dots,x_i;y_i,\dots,y_n,y]$$

$$+\,\omega_{n+1}(x)F[x_0,\dots,x_n,x;y]$$

$$=\frac{\bar{\omega}_{n+1}(y)}{(n+1)!}\sum_{i=0}^{n}\binom{n+1}{i}\frac{\omega_i(x)}{\bar{\omega}_i(y)}\frac{\partial^{n+1}F(\xi_i,\eta_i)}{\partial x^i\partial y^{n-i+1}}$$

$$+\,\frac{\omega_{n+1}(x)}{(n+1)!}\frac{\partial^{n+1}F(\xi,y)}{\partial x^{n+1}},$$

where

$$\xi_i\in I[x_0,x_1,\dots,x_i],\quad i=0,1,\dots,n,$$

$$\eta_i\in I[y_i,y_{i+1},\dots,y_n,y],\quad i=0,1,\dots,n,$$

$$\xi\in I[x_0,x_1,\dots,x_n,x]$$

$$G(x,y)=\sum_{i=2}^{n+1}\frac{\omega_{n+1}(x)}{\omega_i(x)}G[x_n,\dots,x_i,x_{i-1};y]+\frac{\omega_{n+1}(x)}{\omega_1(x)}G[x_n,\dots,x_1,x;y]$$

$$=\sum_{i=2}^{n+1}\frac{\omega_{n+1}(x)}{\omega_i(x)}\left(\sum_{j=0}^{i-2}\bar{\omega}_j(y)G[x_n,\dots,x_{i-1};y_0,y_1,\dots,y_j]\right.$$

$$\left.+\ \bar{\omega}_{i-1}(y)G[x_n,\dots,x_{i-1};y_0,\dots,y_{i-2},y]\right)$$

$$+\,\frac{\omega_{n+1}(x)}{\omega_1(x)}G[x_n,\dots,x_1,x;y]$$

$$= \omega_{n+1}(x) \sum_{i=2}^{n+1} \frac{\bar{\omega}_{i-1}(y)}{\omega_i(x)} G[x_n, \ldots, x_{i-1}; y_0, \ldots, y_{i-2}, y]$$

$$+ \frac{\omega_{n+1}(x)}{\omega_1(x)} G[x_n, \ldots, x_1, x; y]$$

$$= \frac{\omega_{n+1}(x)}{n!} \sum_{i=2}^{n+1} \binom{n}{i-1} \frac{\bar{\omega}_{i-1}(y)}{\omega_i(x)} \frac{\partial^n G(\lambda_i, \theta_i)}{\partial x^{n-i+1} \partial y^{i-1}} + \frac{\omega_{n+1}(x)}{n! \omega_1(x)} \frac{\partial^n G(\lambda, y)}{\partial x^n},$$

where

$$\lambda_i \in I[x_{i-1}, x_i, \ldots, x_n], \quad i = 2, 3, \ldots, n+1,$$

$$\theta_i \in I[y_0, y_1, \ldots, y_{i-2}, y], \quad i = 2, 3, \ldots, n+1,$$

$$\lambda \in I[x_1, x_2, \ldots, x_n, x].$$

Therefore

$$f(x, y) - \mathrm{TNTN}_n(x, y)$$

$$= f(x, y) - [U(x, y)\mathrm{TN}_n^{\mathrm{L}}(x, y) + L(x, y)\mathrm{TN}_n^{\mathrm{U}}(x, y)]$$

$$= U(x, y) \left[\frac{f(x, y)}{L(x, y) + U(x, y)} - \mathrm{TN}_n^{\mathrm{L}}(x, y) \right]$$

$$+ L(x, y) \left[\frac{f(x, y)}{L(x, y) + U(x, y)} - \mathrm{TN}_n^{\mathrm{U}}(x, y) \right]$$

$$= U(x, y) [V(x, y) - \mathrm{TN}_n^{\mathrm{L}}(x, y)] + L(x, y) [V(x, y) - \mathrm{TN}_n^{\mathrm{U}}(x, y)]$$

$$= \frac{U(x, y)}{Q_n^{\mathrm{L}}(x, y)} F(x, y) + \frac{L(x, y)}{Q_n^{\mathrm{U}}(x, y)} G(x, y)$$

$$= \frac{U(x, y)}{Q_n^{\mathrm{L}}(x, y)} \left[\frac{\bar{\omega}_{n+1}(y)}{(n+1)!} \sum_{i=0}^{n} \binom{n+1}{i} \frac{\omega_i(x)}{\bar{\omega}_i(y)} \frac{\partial^{n+1} F(\xi_i, \eta_i)}{\partial x^i \partial y^{n-i+1}} + \frac{\omega_{n+1}(x)}{(n+1)!} \frac{\partial^{n+1} F(\xi, y)}{\partial x^{n+1}} \right]$$

$$+ \frac{L(x, y)}{Q_n^{\mathrm{U}}(x, y)} \left[\frac{\omega_{n+1}(x)}{n!} \sum_{i=2}^{n+1} \binom{n}{i-1} \frac{\bar{\omega}_{i-1}(y)}{\omega_i(x)} \frac{\partial^n G(\lambda_i, \theta_i)}{\partial x^{n-i+1} \partial y^{i-1}} + \frac{\omega_{n+1}(x)}{n! \omega_1(x)} \frac{\partial^n G(\lambda, y)}{\partial x^n} \right].$$

The proof is completed. □

Similarly, one can also obtain the error estimations for the function $f(x, y)$ to be approximated by the other three composite blending rational interpolants $\mathrm{TNNT}_n(x, y)$, $\mathrm{NTTN}_n(x, y)$ and $\mathrm{NTNT}_n(x, y)$.

6. Conclusion

We conclude this paper with a remark.

Remark. Let $\vec{f}_{i,j} \in \mathbb{C}^d$ be d-dimensional vectors to be interpolated at points (x_i, y_j), then the bivariate blending rational interpolants described in Section 4 can be extended to the vector valued case. What is important in this case is to determine how to compute the inverse of a vector $\vec{v} = (v_1, v_2, \ldots, v_d)$. A popular way is to adopt the generalized inverse (or the Samelson inverse) which is defined as (see [1,3])

$$\vec{v}^{\,-1} = \frac{(v_1^*, v_2^*, \ldots, v_d^*)}{\sum_{i=1}^{d} v_i v_i^*}$$

where v_i^* denotes the complex conjugate of v_i. Furthermore, by means of the so-called expansion of matrix into vector (see [6]), one may transplant vector valued blending rational interpolants to matrix valued blending rational interpolants.

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